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THE DIFFRACTION OF A SHOCK WAVE AT THE CURVILINEAR INTERFACE OF TRANSVERSELY ISOTROPIC ELASTIC MEDIA†

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The interaction of a shock wave with the curvilinear interface of transversely isotropic elastic media with different mechanical properties, accompanied by the formation of transmitted and reflected quasi-longitudinal and quasi-transverse shockwaves, is considered. The zeroth approximation of the ray method is used to construct the evolving surface of the shock-wave front and to calculate the values of the discontinuities of the deformation functions at points on this front. The solution of the generalized Snell equations, characterizing the directions of the normals to the wave fronts, departing from the interface, is determined by the method of extension with respect to a parameter together with Newton's algorithm. An example for an ellipsoidal separating surface is considered. Special cases of the transformation of the wave fronts, related to the formation of ray caustics and bifurcations of the front surfaces, and also the occurrence of a total internal reflection effect are established. © 2000 Elsevier Science Ltd. All rights reserved.

The use of the ray method [1–3] in the theory of the propagation of shock waves of different physical kinds enables one to construct the surface of the wave front and to follow the behaviour of the field functions in a certain neighbourhood of it. A systematic description of the history of the development of the ray-series method, as it applies to problems of the propagation and scattering of harmonic and non-stationary isotropic elastic waves, and its well-known generalizations and some mixed problems, which are of theoretical and practical interest, can be found in the monograph [3].

If one has a system of wave beams, one can formulate the problem of separating their families, having common envelopes (caustics), on which the rays are focused and the field intensity increases without limit, and also along which the particular features of the wave front move. In geometrical optics the caustics and the particular features of the wave fronts are classified using the theory of singularities of differentiable mappings – catastrophe theory [1, 4–6].

Problems of the dynamic theory of elasticity for anisotropic elastic media have been investigated to a much lesser extent than similar problems for isotropic media, since effective methods, developed for the case of isotropy, do not lend themselves to a simple extension to anisotropic systems because of the complexity of the latter's properties. Plane monochromatic waves have been most completely investigated here, and an analysis of the propagation and diffraction of such waves are analysed in detail in [2, 7]. A method of determining the form of the surface of the elastic wave front, excited by a source of arbitrary form in an unbounded anisotropic medium, including the solution of the Cauchy problem for the equation of the characteristics was proposed in [8]. A considerable contribution to the development of the ray method, as it applies to problems of propagation and diffraction of elastic waves in anisotropic media, was made by the Leningrad school of mathematicians [9, 2, etc.]. A ray theory of shock was recently proposed [10], which enables the problem of the interaction of an elastic striker and an elastic target to be effectively solved; other problems are also solved. A review of the latest achievements in this area can be found in [11].

In problems of the dynamics of anisotropic elastic media, due to the fact that their behaviour differs by the presence of different types of polarized waves, the phase velocities of which depend on the direction of motion, the rays cease to be orthogonal surfaces of the phase fronts and, in general, additional difficulties arise. Thus, the denominators of the corresponding Snell's equations, describing the kinematics of the interaction of the shock waves with the interfaces of elastic media with different mechanical properties, cease to be constants and the equations themselves become extremely non-linear, and the method of solving them also becomes more complicated. The particular solutions of these

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equations are related to the bifurcations of the wave fronts, which are formed as a result of reflection-refraction effects.

1. RESOLVENTS

The motion if an elastic anisotropic medium, characterized by a tensor of the elasticity constants $C_{ik,pq}$ and a density ρ, is described by the following equations in a rectangular Cartesian system of coordinates $Ox_1x_2x_3$

$$\sum_{k,p,q=1}^{3} \lambda_{ik,pq} \frac{\partial^2 u_q}{\partial x_k \partial x_p} - \frac{\partial^2 u_i}{\partial t^2} = 0, \quad i = 1,2,3 \tag{1.1}$$

Here $\lambda_{ik,pd} = C_{ik,pq}$; and u_1, u_2, and u_3 are the components of the elastic displacement vector.

We will represent the solutions of system (1.1) in the form of a plane monochromatic wave with wave number k and phase velocity υ. Its fronts are surfaces of constant phase $\mathbf{nr} - vt$ = const, moving with a velocity $\mathbf{v} = \upsilon \cdot \mathbf{n}$ and coinciding locally with areas perpendicular to the unit vector $\mathbf{n}$.

The polarization vector $\mathbf{A}$ of the wave and its phase velocity υ for the chosen direction $\mathbf{n}$ are found from the homogeneous system of algebraic equations [2, 6]

$$\sum_{p,q=1}^{3} \lambda_{ik,pq} n_k n_p A_q - \upsilon^2 A_i = 0, \quad i = 1,2,3 \tag{1.2}$$

the matrix of whose coefficients possesses the properties of symmetry and positive definiteness.

From the condition for non-trivial solutions of this system to exist we obtain an equation of the third degree in υ^2 with kernels $\upsilon_r^2(\mathbf{n})$. For each value of $\upsilon_r^2(\mathbf{n})$ (r = 1, 2, 3) we construct a system (1.2) by means of which we calculate the orthogonal components of the polarization vector $\mathbf{A}^{(r)}$ of a wave moving in the direction n with phase velocity $\upsilon_r(\mathbf{n})$.

The surface of the shock-wave front can be represented by the relation

$$\tau(x_1, x_2, x_3) - t = 0 \tag{1.3}$$

in which the function τ must satisfy the first-order partial differential equation [7]

$$\sum_{i,k,p,q=1}^{3} \lambda_{ik,pq} p_k p_p A_q^{(r)} A_i^{(r)} = 1 \tag{1.4}$$

which generalizes the eikonal equation in geometrical optics to the case of anisotropic elastic media.

The quantities p_k (k = 1, 2, 3) in (1.4) are the components of the refraction vector $p_k \equiv \partial\tau/\partial x_k = n_k/\upsilon_r(\mathbf{n})$ (k = 1, 2, 3).

To construct wave front (1.3) we need to obtain solutions of Eq. (1.4) which, using the method of characteristics, can be reduced to the following system of ordinary differential equations

$$\frac{dx_k}{d\tau} = \xi_k = \sum_{i,p,q=1}^{3} \lambda_{ik,pq} p_p A_q^{(r)} A_i^{(r)}, \quad \frac{dp_k}{d\tau} = 0, \quad k = 1,2,3 \tag{1.5}$$

The first group of these equations describes the propagation of the wave along a ray with ray velocity $\xi = \xi^{(r)}(\mathbf{n}, x_k)$, which, for uniform media are rectilinear, but in general, not orthogonal to the phase front. Points on the surface of the front, at which the determinant of the matrix

$$\left\| \sum_{i,p,q=1}^{3} \lambda_{ik,pq} A_q^{(r)} A_i^{(r)} \right\|, \quad k = 1,2,3$$

are the coefficients of the right-hand side of this system vanish, are bifurcation points, since, in a small neighbourhood of them, two or more directions of the rays may correspond to one direction of the vector of the normal $\mathbf{n}$. Hence, these points either on caustics or at focal points of the rays.

The system of rays and wave fronts constructed using (1.5) enable us to proceed to determine the wave intensity in the neighbourhood of the wave front. To do this we use the zeroth term of the series-expansion of the solution of system (1.1) along a ray

$$u_q = \sum_{m=0}^{\infty} u_q^{(m)}(x_1, x_2, x_3) f_m[t - \tau(x_1, x_2, x_3)], \quad q = 1,2,3 \tag{1.6}$$

where the functions f_m, which satisfy the relations $f'_m(y) = f_{m-1}(y)$, are assumed to have breaks in continuity in the derivatives, for example, of the order of $n + 2$ [2].
In this case the vector $\mathbf{u}^{(0)}$ is found from the homogeneous system of equations

$$\sum_{k,p,q}^{3} \lambda_{ik,pq} p_k p_p u_q^{(0)} - u_i^{(0)} = 0, \quad i = 1,2,3 \tag{1.7}$$

the solution of which can be represented in the form [2]

$$u_q^{(0)} = \frac{c_0(\alpha, \beta) A_q^{(r)}(\tau, \alpha, \beta)}{\sqrt{J(\tau, \alpha, \beta)}} f_0[t - \tau(x_1, x_2, x_3)], \quad q = 1,2,3 \tag{1.8}$$

where τ, α, β is a system of ray coordinates, while the functional determinant $J = \partial(x_1, x_2, x_3)/\partial(\tau, \alpha, \beta)$ of the conversion of the ray coordinates into Cartesian coordinates is a measure of the divergence of the rays in the ray tube.

2. METHOD OF SOLUTION

The relations given above enable us to follow the evolution of the shock-wave front and to calculate the values of the discontinuities of the field functions on its surface outside the interface of anisotropic elastic media with different properties. We will consider the case of the interaction of an axisymmetric shockwave, emitted from a source C, with an ellipsoidal surface of revolution G at the interface of two transversely isotropic elastic media (Fig. 1). Suppose the axes of symmetry of infinitely high order of the elastic properties of their materials and the axis of revolution of the ellipsoid are parallel to one another and that the problem is axisymmetrical. We will use the "locally plane approximation" [2], according to which, at the point where the wave is incident on the interface G the rays of the incident, reflected and refracted waves, and also their polarization vectors lie in one plane, containing the ray

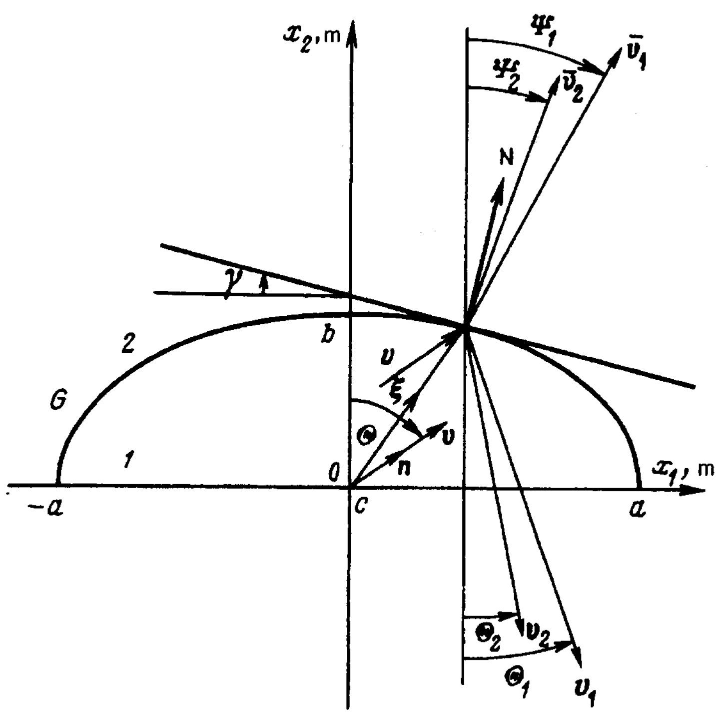

Fig. 1

of the incident wave and the normal to the surface. This enables us to introduce the angles Θ_ν and Ψ_μ (μ, ν = 1, 2) at which the reflected and refracted waves leave the boundary G, and we will use the generalized Snell's law, expressed by the equations

$$\frac{\sin(\Theta-\gamma)}{\upsilon(\Theta)}=\frac{\sin(\Theta_\nu+\gamma)}{\upsilon_\nu(\Theta_\nu)}=\frac{\sin(\Psi_\mu-\gamma)}{\bar{\upsilon}_\mu(\Psi_\mu)},\quad \nu,\mu=1,2 \tag{2.1}$$

Here Θ is the angle between the direction of the wave normal of the wave incident on medium 1 and the Ox_2 axis, and Θ_ν, Ψ_μ (ν, μ = 1, 2) are the angles between the direction of the wave normal and the Ox_2 axis of the waves reflected in medium 1 and penetrating into medium 2, respectively (Fig. 1). The values of the subscripts $\nu = \mu = 1$ correspond to quasi-longitudinal waves, the values $\nu = \mu = 2$ correspond to quasi-transverse waves and γ is the slope of the tangent to the elliptical interface.

The difference between relations (2.1) and the form of Snell's law for isotropic media is due to the dependence of the denominator $\upsilon_\nu(\Theta_\nu)$, $\upsilon_m(\Psi_\mu)$ on the corresponding angles Θ_ν, Ψ_μ, and implicitly on the angle of incidence Θ. Hence, in order to obtain the values of the angles Θ_ν, Ψ_μ (ν, μ = 1, 2), corresponding to a given Θ, we need to solve non-linear system of equations (2.1). To do this we will use Newton's method together with the method of the continuation of the solution with respect to a parameter [12, 13]. A consequence of this is that in system (2.1) the angle γ is variable and when constructing its solutions $\Theta_\nu(\Theta)$, $\Psi_\mu(\Theta)$ we will vary it also. We will choose the variable Θ as the leading parameter.

Consider the first equation of system (2.1). Suppose that, for a certain value of $\Theta = \Theta^{(n)}$, the angles $\Theta_\nu^{(n)}$ and $\gamma^{(n)}$ have been determined. We give a small increment $\Delta\Theta^{(n)}$ to $\Theta^{(n)}$. The increment $\Delta\gamma^{(n)}$ corresponding to it is obtained as the difference of the angles $\gamma^{(n+1)} - \gamma^{(n)}$ corresponding to the values $\Theta^{(n+1)}$, $\Theta^{(n)}$ for the chosen profile of the interface. In order to obtain the increments $\Delta\Theta_\nu^{(n)}$ we will write the following equation, which follows from (2.1):

$$\begin{aligned}&\sin(\Theta^{(n)}-\gamma^{(n)}+\Delta\Theta^{(n)})\upsilon_\nu(\Theta_\nu^{(n)}+\gamma^{(n)}+\Delta\Theta_\nu^{(n)})-\\&-\sin(\Theta_\nu^{(n)}+\gamma^{(n)}+\Delta\Theta_\nu^{(n)})\upsilon(\Theta^{(n)}-\gamma^{(n)}+\Delta\Theta^{(n)})=0\end{aligned}$$

Separating its linear part, we obtain

$$\begin{aligned}\Delta\Theta_\nu^{(n)}\approx&\{[\sin(\Theta_\nu^{(n)}+\gamma^{(n)})\partial\upsilon(\Theta^{(n)})/\partial\Theta-\cos(\Theta^{(n)}-\gamma^{(n)})\upsilon_\nu(\Theta_\nu^{(n)})]\Delta\Theta^{(n)}+\\&+[\cos(\Theta_\nu^{(n)}+\gamma^{(n)})\upsilon(\Theta^{(n)})-\cos(\Theta_\nu^{(n)}-\gamma^{(n)})\partial\upsilon_\nu(\Theta_\nu^{(n)})/\partial\Theta]\Delta\gamma^{(n)}\}\times\\&\times[\sin(\Theta^{(n)}-\gamma^{(n)})\partial\upsilon_\nu(\Theta_\nu^{(n)})/\partial\Theta_\nu-\cos(\Theta_\nu^{(n)}+\gamma^{(n)})\upsilon(\Theta^{(n)})]^{-1},\quad \Delta\gamma=f(\Delta\Theta)\end{aligned} \tag{2.2}$$

Calculating $\Delta\Theta_\nu^{(n)}$, we obtain the kinematic parameters of an element of the surface of the wave front $\Theta^{(n+1)} = \Theta^{(n)} + \Delta\Theta^{(n)}$, $\Theta_\nu^{(n+1)} = \Theta_\nu^{(n)} + \Delta\Theta_\nu^{(n)}$, $\upsilon_\nu(\Theta_\nu^{(n+1)})$. Linearizing Eqs (2.1) in the neighbourhood of its state, we determine $\Delta\Theta^{(n+1)}$ corresponding to $\Delta\Theta^{(n+1)}$, etc. It is necessary to bear in mind here that, since relation (2.2) is approximate, the error in calculating $\Delta\Theta_\nu^{(m)}$ will increase as m increases. Hence, to compensate for the inaccuracy it is necessary, as is done in Newton's method, to add to the right-hand side of (2.2) the discrepancy of Eqs. (2.1), taken with the opposite sign. We finally obtain the computational scheme (the dots denote the right-hand side of the approximate equation (2.2))

$$\Delta\Theta_\nu^{(n)}\approx\ldots+\sin(\Theta_\nu^{(n)}+\gamma^{(n)})\upsilon(\Theta^{(n)})-\sin(\Theta^{(n)}-\gamma^{(n)})\upsilon_\nu(\Theta_\nu^{(n)}) \tag{2.3}$$

which combines the method of continuation of the solution with respect to a parameter and Newton's method, the accuracy of which increases as $\Delta\Theta^{(n)}$. To calculate the derivative $\partial\upsilon/\partial\Theta$ we need to differentiate the left-hand side of Eq. (1.2) with respect to Θ and equate the derivative to zero.

We will denote by B the matrix of the elements

$$b_{iq}=\sum_{k,p=1}^{3}\lambda_{ik,pq}n_k n_p-\upsilon^2\delta_{iq}$$

the cofactors of which B_{iq} (i, q = 1, 2, 3) Then, since condition (1.2) remains the same when Θ is varied, we have $\partial|B|/\partial\Theta = 0$, or

$$\sum_{i=1}^{3}\sum_{q=1}^{3}\frac{\partial b_{iq}}{\partial\Theta}B_{iq}=0$$

Substituting into this the equation

$$\frac{\partial b_{iq}}{\partial\Theta}=\sum_{k,p=1}^{3}\lambda_{ik,pq}\left(n_k\frac{\partial n_p}{\partial\Theta}+n_p\frac{\partial n_k}{\partial\Theta}-2\upsilon\frac{\partial\upsilon}{\partial\Theta}\delta_{iq}\right)$$

we obtain an equation from which we find $\partial\upsilon/\partial\Theta$. Carrying out these calculations in the neighbourhood of $\Theta=\Theta^{(n)}$, $\upsilon=\upsilon(\Theta^{(n)})$ we obtain $\partial\upsilon/\partial\Theta^{(n)}$. We can similarly find $\partial\upsilon_\nu(\Theta_\nu^{(n)})\partial\Theta_\nu$.

It is possible to carry out calculations using scheme (2.3) when there is a certain initial state $\Theta^{(0)}$, $\upsilon(\Theta^{(0)})$, $\Theta_\nu^{(0)}$, $\upsilon_\nu(\Theta_\nu^{(0)})$ $\gamma^{(0)}$. For this case, in accordance with (2.1), it is convenient to choose $\Theta^{(0)}=0$, $\upsilon(0)$, $\Theta_\nu^{(0)}=0$, $\upsilon_\nu(0)$, $\gamma^{(0)}=0$. Formula (2.3) enables us to obtain the unique increment $\Delta\Theta_\nu$ for a value of the angle of incidence Θ for which the denominator on the right-hand side of (2.3) is non-zero. Hence, the equation

$$\sin(\Theta^{(n)}-\gamma^{(n)})\partial\upsilon_\nu(\Theta_\nu^{(n)})/\partial\Theta_\nu-\cos(\Theta_\nu^{(n)}+\gamma^{(n)})\upsilon(\Theta^{(n)})=0 \tag{2.4}$$

is the condition for the bifurcation of the solution. To continue the solution through this state we must add terms of the second order (and if necessary of the third, etc. order) in (2.3) [12].

Condition (2.4) of the possible non-uniqueness of the solutions of system (2.1) corresponds to the convergence (tangency) and intersection of the reflected and refracted rays after the incident rays interact with the interface G, while the set of such critical situations is related to the formation of an envelope of a family of rays -- caustics. The caustics may give rise to the formation of geometrical singularities on the surfaces of the reflected and refracted wave fronts as a result of the interaction of a regularly reflected wave front even with a plane boundary G.

Since the singularities of the wave front are generated on the caustics, it will also be focused on the caustics, accompanied by the vanishing of the functional determinant J in (1.8) and an unlimited increase in the field intensity at points of geometrical singularities. The wave phase is also reversed on the caustics [1].

The most general features of the bifurcations of the caustics and wave fronts were investigated previously in [1] by the methods of the theory of discontinuities of differentiable mappings. The method proposed in this paper enables one to carry out a computer simulation for anisotropic elastic media.

3. RESULTS OF INVESTIGATIONS

Using the approach described above we solved the problem of the interaction of the wave fronts of quasi-longitudinal and quasi-transverse shock waves with the ellipsoidal interface G of two transversely isotropic elastic media. The waves are generated in medium 1 by a spherical source of small radius, which is at the geometrical centre of the separating ellipsoidal surface of revolution, the axis of which coincides with the Ox_2 axis of symmetry of the elasticity parameters of the media.

Since the elastic properties of the transversely isotropic media are characterized by only five irreducible parameters, we will represent the components $C_{ik,pq}$ of their elasticity constants tensors in the form of a square matrix [2].

$$\|C_{\alpha\beta}\|=\begin{Vmatrix}L & O\\ O & M\end{Vmatrix};\quad L=\begin{Vmatrix}\lambda+2\mu & \lambda-l & \lambda\\ \lambda-l & \lambda+2\mu-p & \lambda-l\\ \lambda & \lambda-l & \lambda+2\mu\end{Vmatrix} \tag{3.1}$$

$$M=\mathrm{diag}(\mu-m,\mu,\mu-m)$$

the correspondence between the elements of which and the components of the tensor $C_{ik,pq}$ are established from the scheme

$$(11)\leftrightarrow 1,\ (22)\leftrightarrow 2,\ (33)\leftrightarrow 3,\ (23)=(32)\leftrightarrow 4,$$
$$(31)=(13)\leftrightarrow 5,\ (12)=(21)\leftrightarrow 6 \tag{3.2}$$

Hence, the three parameters l, m, and p in (3.1) and (3.2) characterize the difference between the media considered and an isotropic medium with Lame parameters λ and μ.

Suppose a normal pressure is applied to a spherical cavity C of medium 1, which initiates not only a quasi-longitudinal shock wave (qP), as in isotropic media, but also a quasi-transverse shock wave (qS), the surfaces of the wave fronts of which possess axial symmetry [2]. Because of this the intensity of the quasi-transverse wave, polarized orthogonally to the first two, will be zero. Hence also, after interaction with the interface G, only two forms of axisymmetrical reflected and refracted waves will be formed, polarized in the plane of the axial section.

We will denote by the subscripts 1 and 2 the parameters of waves which propagate in the first and second medium respectively, the parameters of the waves before and after interaction with the surface G will be identified by subscripts minus and plus.

The values of the mechanical constants, characteristic of the corresponding isotropic bodies, were chosen to be those of dolomite (for medium 1) and sandstone (for medium 2): $\lambda_1 = 4.972 \times 10^{10}$ Pa, $\mu_1 = 3.906 \times 10^{10}$ Pa, $\rho_1 = 2.650 \times 10^3$ kg/m^3, $\lambda_2 = 3.409 \times 10^9$ Pa, $\mu_2 = 1.364 \times 10^{10}$ Pa and $\rho_2 = 2.760 \times 10^3$ kg/m^3. The values of l m and p, which disturb the isotropy properties, were varied, and the calculations were carried out for various combinations of their values. In Fig. 2(a) and (b) we show the results for $l_i = 0.1\ \lambda_i$ $m_i = 0.3\ \mu_i$ and $p_i = 0.5\ (\lambda_i + 2\mu_i)$ $(i = 1, 2)$.

Figure 2(a) illustrates the form the axial sections of the wave fronts of the incident shock waves qP_{1-} (1), the reflected shock waves qP_{1+} (2), qS_{1+} (3) and the refracted shock waves qP_{2+} (4), qS_{2+} (5) produced by the qP_{1-}-wave issuing from a point source, when the angle between the normal to the incident wave front and the Ox_2 axis is $\Theta_-^{(f)} = 85°$. The small circles on these graphs indicate the intersections of the line of the wave front with rays corresponding to the incident rays with a difference in the angles of incidence between the normals $\mathbf{n}$ at the source C of $\Delta\Theta_0 = 4°$. At points

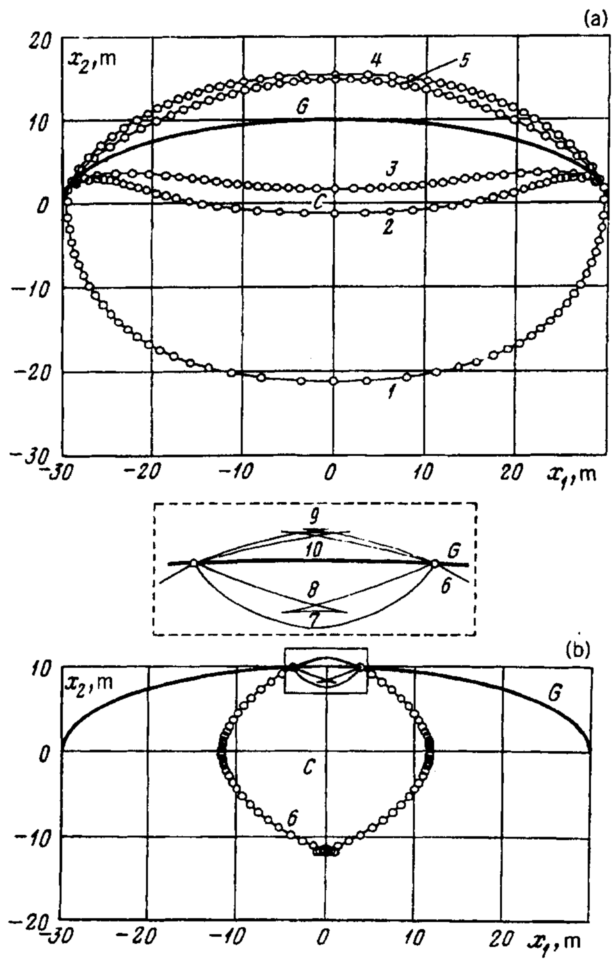

Fig. 2

where they are bunched together the divergence of the rays decreases, and in these zones of the surface of the wave front the stress intensity is increased. For the case considered a distinctive feature is the absence of singularities on all the types of original and formed wave fronts.

The nature of the interaction of the quasi-transverse qS_{1-}-wave with the interface G is shown in Fig. 2(b). It should be noted that the section of its wave front, up to the time it is incident on the interface G (curve 6), has symmetrically situated swallowtail type singularities, which are also retained on the wave fronts of the qP_{1+}-, qS_{1+}-, qP_{2+}-, qS_{2+}-waves formed (curves 7–10, respectively).

In Fig. 3 we show a sequence of wave fronts of the qS_{1+}-wave as its leaves the spherical surface of the source C (because of symmetry we only show the region $x_2 \geqslant 0$). Initially the surface has a smooth profile, and then cuspidal points appear on it. The curves mapped out by these points as the wave front is transformed are caustics. On these (assuming ideal elasticity) the field intensity increases without limit, and on passing through them the phase of the qS_{1-}-wave is reversed [1].

As a result of the interaction of the initial qP_{1-}-wave and qS_{1-}-wave with the surface G reflected (qP_{1+}-, qS_{1+}-) waves and refracted (qP_{2+}-, qS_{2+}-) waves are formed (Fig. 2a, curves 2–5 and Fig. 2b, curves 7–10). The directions of their rays were determined using algorithm (2.3). The solution of Eqs (2.1) for the case when a qS_{1-}-wave is incident is shown in Fig. 4 in the form of graphs of $\Theta_{\nu+}(\Theta_{2-})$ (the continuous curves) and $\Psi_{\mu+}(\Theta_{2-})$ (the dashed curves), where curves 1 and 2 correspond to $\upsilon = 1, 2$ and curves 3 and 4 correspond to $\mu = 1, 2$. We can conclude from an analysis of these curves that for a quasi-longitudinal reflected wave qP_{1+} (curve 1) the value of the angle of incidence $\Theta_{2-} = 29.7°$ is the limiting value, after which the wave ceases to be a shock wave and the wave theory is inapplicable . In the small semineighbourhood $\Theta_{2-} \leqslant 29.7°$ the rays of the wave front of shock wave qP_{1+} bunch together.

The form of curve 2 confirms that even for similar qS_{1-}- and qS_{1-}-waves the angle of incidence is not equal to the angle of reflection (as it would be in the case of isotropic elastic media [13]). At the point where this curve reaches a minimum two values of the angle of incidence of the qS_{1-}-wave correspond to a single value of the angle of reflection of the qS_{1+}-wave. Consequently, this point is a cuspidal point for the wave front of the qS_{1+}-wave. Condition (2.5) is satisfied at this point and bifurcational formation of a caustic occurs.

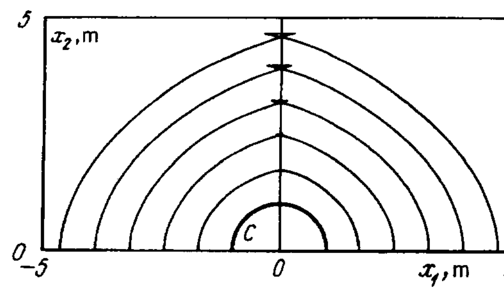

Fig. 3

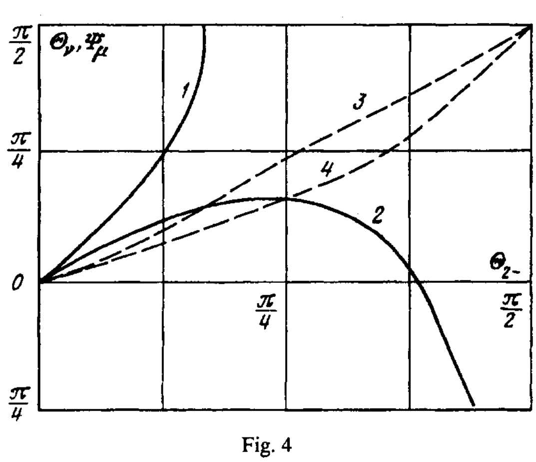

Fig. 4

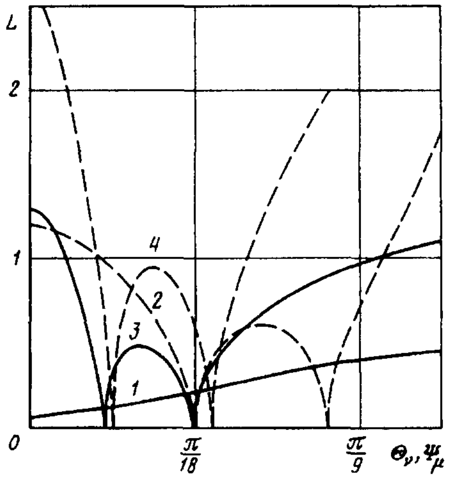

Fig. 5

Figure 5 illustrates the geometrical divergences of the beams of rays [2]

$$L(\Theta) = \sqrt{|J(\Theta)|/\xi(\Theta)} \tag{3.3}$$

as a function of the angle $\Theta_\nu(\Psi_\mu)$ between the normal to the corresponding surface of the wave front and the Ox_2 axis for waves qP_{1+}, qS_{1+}, qP_{2+}, qS_{2+} (curves 1–4, respectively). Since the quantity $\sqrt{J}$ defined by (3.3) occurs in the denominator of Eq. (1.8), the values of the angles $\Theta_\nu(\Psi)$ for which curves 1–4 are incident on the abscissa axis, are singular. They correspond to the formation of singularities on the surface of the corresponding wave fronts. The method of calculating the geometrical divergence, based on the ray method, leads to fairly accurate results in regions of the ray field remote from the singular points and caustics.

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